MATH 3801: Logic

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Sample Examination Questions

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1. (a) Define the set of formulae, $\mathcal{L}$, in the first order predicate language.

   (b) Define the degree of a formula.

   (c) Define the weight of an element of $\mathcal{L}_{string}$.

   (d) Prove that, if α is a formula, then the weight of α is -1.

   (e) Consider the following elements of $\mathcal{L}_{string}$, where x, y, z are variables, P is a unary predicate, and Q is a binary predicate:

   (i) $\neg\forall x \Rightarrow PxQxy$

   (ii) $\forall xz \Rightarrow PzQxz$

   (iii) $\Rightarrow\Rightarrow PxPy\neg \Rightarrow QxyQyx$

   Compute the weight of each of the above strings.

   For each string above, determine whether or not it is a formula, justifying your answer.

2. (a) Define the set of formulae, $\mathcal{L}$, in the first order predicate language.

   (b) Define the degree of a formula.

   (c) Define the weight of a string.

   (d) Prove that, if α is a formula, then no proper initial segment of α is a formula. (You may assume that any formula has weight -1).

   (e) Consider the following strings, where x, y, z are variables, P is a unary predicate, and Q is a binary predicate:

   (i) $\neg\forall x\forall yQxy$

   (ii) $\forall x \Rightarrow Pz \Rightarrow Qxz$

   (iii) $\Rightarrow \neg\forall xPxQxy$

   For each string above, determine whether or not it is a formula, justifying your answer.

3. (a) Describe the symbols of the first order predicate language.

   (b) Define the set of formulae, $\mathcal{L}$, in the first order predicate language.

   (c) Define the degree of a formula.

   (d) For each string below, determine, using the definition of $\mathcal{L}$, if the string is an element of $\mathcal{L}$ (where x, y, z are variables, P is a unary predicate, and Q is a binary predicate):

      (i) $\forall y P y z$

      (ii) $\forall x \neg Q y z$

      (iii) $\Rightarrow P x P y \neg \forall x P x$

      (iv) $\Rightarrow P x \Rightarrow Q y z P x$

   (e) Consider the following formulae, written in the conventional functional first order predicate language, $\mathcal{L}_{math}$:

      (i) $(\neg\neg P x) \Rightarrow P x$

      (ii) $\neg(P x \vee Q y z)$

      (iii) $\exists x((\neg P x) \Rightarrow Q x y)$

      (iv) $\neg((Q x y \Rightarrow Q y z) \Rightarrow (Q y z \Rightarrow Q x y))$

   Convert each of the above to a formula in $\mathcal{L}$.

4. (a) Describe the symbols of the first order predicate language, and the set $\mathcal{L}$ of first order predicate formulae.

   (b) Convert each of the following formulae of $\mathcal{L}_{math}$ to a formula in $\mathcal{L}$ (below, x, y are variables, P is a unary predicate, and Q is a binary predicate):

      (i) $P x \vee Q y y$

      (ii) $Q y z \wedge Q x y$

      (iii) $\exists x P x$

      (iv) $P x \Leftrightarrow P y$

      (v) $\neg((\exists x P x) \Rightarrow (Q x y \vee Q y x))$

   (c) Define the degree of a formula.

   (d) Define the weight of a string.

   (e) Show that any formula has weight -1.

5. (a) Describe the symbols of the language of propositional logic.
   (b) Define $\mathcal{L}_0$, the set of propositions.
   (c) Write down the basic rules for propositional semantic tableaux.
   (d) Use the semantic tableaux method to determine whether or not each of the following holds (where α, β, γ are primitive propositions):
      (i) $\models (\neg\alpha \Rightarrow \neg\beta) \Rightarrow ((\neg\alpha \Rightarrow \beta) \Rightarrow \alpha)$
      (ii) $\{\alpha \Rightarrow (\beta \Rightarrow \gamma)\,,\, \beta\} \models (\alpha \Rightarrow \gamma)$
      (iii) $\models (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$

6. (a) Give the definition of a valuation on the set $\mathcal{L}_0$ of propositions.
   (b) State what it means for a proposition α to be true for a valuation v.
   (c) Suppose that $S \subset \mathcal{L}_0$, and $\alpha \in \mathcal{L}_0$. State what $S \models \alpha$ means.
   (d) State what it means to say that a proposition α is a tautology.
   (e) Use the semantic tableaux method to determine whether or not each of the following propositions is a tautology (where α, β, γ are primitive propositions). If a proposition is not a tautology, describe a valuation for which it fails to be true.
      (i) $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow (\alpha \Rightarrow \gamma)$
      (ii) $(\alpha \Rightarrow (\beta \vee \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \vee (\alpha \Rightarrow \gamma))$
      (iii) $((\alpha \wedge \beta) \Rightarrow \gamma) \Rightarrow ((\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma))$

7. (a) Define what it means to have a proof of a proposition α from a set of propositions S, giving also the set of axioms and the rule of deduction.
   (b) State what it means for a proposition α to be a theorem.
   (c) State and prove the Deduction Theorem for propositional logic.
   (You may assume the validity of the following theorem: $\vdash (\alpha \Rightarrow \alpha)$)
   (d) Use the Deduction Theorem to show the following:
      (i) $\{\alpha \Rightarrow (\beta \Rightarrow \gamma)\,,\, \beta\} \vdash (\alpha \Rightarrow \gamma)$
      (ii) $\{\alpha \Rightarrow \beta\,,\, \beta \Rightarrow \gamma\} \vdash (\alpha \Rightarrow \gamma)$

8. (a) Give a direct proof of the following theorem (for a proposition α):

$$\vdash (\neg\alpha \Rightarrow \neg\alpha)$$

(b) State the Deduction Theorem for propositional logic.

(c) Use the Deduction Theorem to show the following (for propositions α, β):

$$\vdash (\neg\alpha) \Rightarrow (\alpha \Rightarrow \beta)$$
$$\vdash (\neg\neg\alpha) \Rightarrow \alpha$$

(You may assume the validity of the theorem given in part (a) above.)

(d) Use the Deduction Theorem to show the following (for a proposition α):

$$\vdash \alpha \Rightarrow (\neg\neg\alpha)$$

(You may assume the validity of the theorems given in parts (a), (c) and (d) above.)

(e) State what it means for a set S of propositions to be consistent.

(f) Prove that, if S is a consistent set of propositions and α is any proposition, then at least one of $S \cup \{\alpha\}$ or $S \cup \{\neg\alpha\}$ is consistent.
(You may assume the validity of the theorems given in parts (a) and (c) above.)

9. (a) Give the definition of a valuation on the set $\mathcal{L}_0$ of propositions.

(b) State what it means to say that a proposition α is a tautology.

(c) Define what it means to have a proof of a proposition α from a set of propositions S, giving also the set of axioms and the rule of deduction.

(d) State what it means to say that a proposition α is a theorem.

(e) State the Deduction Theorem for propositional logic.

(f) Consider a proposition π, which has the following form (for primitive propositions α, β, γ):

$$((\alpha \Rightarrow \beta) \Rightarrow \gamma) \Rightarrow (\beta \Rightarrow \gamma)$$

Use a truth table to show that π is a tautology.
Use the semantic tableaux method to show that π is a tautology.
Use the Deduction Theorem to show that π is a theorem.

10. (a) Give the definition of a valuation on the set $\mathcal{L}_0$ of propositions.

(b) Use the semantic tableaux method to show that the following holds (where α, β, γ are primitive propositions):

$$\{\alpha \Rightarrow (\beta \Rightarrow \gamma)\,,\, \beta\} \models (\alpha \Rightarrow \gamma)$$

(c) State and prove the Deduction Theorem for propositional logic.

(d) Consider the following syntactic implication:

$$\{\alpha \Rightarrow (\beta \Rightarrow \gamma)\,,\, \beta\} \vdash (\alpha \Rightarrow \gamma)$$

Give a direct proof of the above syntactic implication.

Give a proof that uses the Deduction Theorem, of the same implication.

11. (a) Give the definition of a valuation on the set $\mathcal{L}_0$ of propositions.

(b) State what it means to say that a proposition α is a tautology.

(c) Define what it means to have a proof of a proposition α from a set of propositions S, giving also the set of axioms and the rule of deduction.

(d) Use the semantic tableaux method to verify that each of the axioms that may be used in a proof is a tautology.

(e) State and prove the Soundness Theorem for propositional logic.

12. (a) State what it means for a set of propositions to be consistent.

(b) Let S be a set of propositions ($S \subset \mathcal{L}_0$). Prove that, if S is consistent, then S has a model.

(You may assume that, if S is a consistent set of propositions and α is any proposition, then at least one of $S \cup \{\alpha\}$ or $S \cup \{\neg\alpha\}$ is consistent.)

(c) State the Completeness Theorem for propositional logic.

(d) State and prove two consequences of the Completeness Theorem for propositional logic.

13. (a) Define an $\mathcal{L}(\Pi, \Omega)$-structure, where Π is a given set of predicate symbols and Ω is a given set of functional symbols.

(b) (i) Define the set of terms in a given first order predicate language $\mathcal{L}(\Pi, \Omega)$.

(ii) Give the definition of a closed term.

(iii) State what it means for a formula to be a sentence.

(iv) State what a theory is.

(c) Given an $\mathcal{L}(\Pi, \Omega)$-structure U, define the interpretation of a closed term in U, and the interpretation of a sentence in U.

(d) Describe a theory in a suitably defined first order predicate language $\mathcal{L}(\Pi, \Omega)$, such that a structure U is a (normal) model of the theory if and only if U is a group of order 3.

14. (a) Define an $\mathcal{L}(\Pi, \Omega)$-structure, where Π is a given set of predicate symbols and Ω is a given set of functional symbols.

(b) Describe a theory in a suitably defined first order predicate language $\mathcal{L}(\Pi, \Omega)$, such that a structure U is a (normal) model of the theory if and only if U is a ring of size 3.

(c) Suppose that S is a set of sentences in a first order predicate language. State what it means to say that S is consistent.

(d) State and prove the Compactness Theorem for first order predicate logic.

(You may assume the following form of the Completeness Theorem for first order predicate logic: If S is a set of sentences in a first order predicate language, then S is consistent if and only if S has a model.)

15. (a) Define what it means to have a proof of a first order predicate formula α from a set of first order predicate formulae T, giving also the set of axioms and the rules of deduction.

(b) State and prove the Deduction Theorem for first order predicate logic.

(c) Describe the theory P of (strong) Peano arithmetic, in a suitably defined first order predicate language $\mathcal{L}(\Pi, \Omega)$.

16. (a) State the Completeness Theorem for first order predicate logic.

(b) State the Compactness Theorem for first order predicate logic.

(c) State and prove the Upper Löwenheim-Skolem Theorem for first order predicate logic.

(d) Describe the theory of (strong) Peano arithmetic, in a suitably defined first order predicate language $\mathcal{L}(\Pi, \Omega)$.

(e) Is every (normal) model of this theory isomorphic to the natural numbers? Justify your answer.

17. (a) Define the notion of a register machine, giving also a description of what a program is and of the types of instructions associated to the states of a program.
    (b) State what it means for a function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ to be computable.
    (c) Show that the following functions are computable:
        (i) $f : \mathbb{N}_0 \to \mathbb{N}_0\,, \quad f(m) = 0$
        (ii) $f : \mathbb{N}_0^2 \to \mathbb{N}_0\,, \quad f(m,n) = n$
        (iii) $f : \mathbb{N}_0 \to \mathbb{N}_0\,, \quad f(m) = 2m$

18. (a) Define the set of recursive partial functions.
    (b) Show that the following functions are recursive:
        (i) $f : \mathbb{N}_0^2 \to \mathbb{N}_0\,, \quad f(m,n) = m+n$
        (ii) $f : \mathbb{N}_0^2 \to \mathbb{N}_0\,, \quad f(m,n) = mn$
        (iii) $f : \mathbb{N}_0^2 \to \mathbb{N}_0\,, \quad f(m,n) = m^n$
    (c) Show that the the following function is recursive:

$$f : \mathbb{N}_0 \to \mathbb{N}_0\ , \quad f(n) = 0 \text{ if } n \text{ is odd}\ , \ f(n) = 1 \text{ if } n \text{ is even}$$

    (You may assume that the functions given in part (b) are recursive, but are required to verify that any other functions you use are recursive.)

19. (a) Define the notion of a register machine, giving also a description of what a program is and of the types of instructions associated to the states of a program.
    (b) State what it means for a function $f : \mathbb{N}_0^k \to \mathbb{N}_0$ to be computable.
    (c) Prove that the composition of computable (partial) functions leads to a computable (partial) function.
    (d) Give a way of uniquely encoding each instruction that may be associated to a state of a program, by a natural number.
    (e) Give a way of uniquely encoding each program, by a natural number.

20. (a) Define the set of recursive partial functions.
    (b) Show that the following functions are recursive:
        (i) $f : \mathbb{N}_0^2 \to \mathbb{N}_0\,, \quad f(m,n) = m+n$
        (ii) $f : \mathbb{N}_0^2 \to \mathbb{N}_0\,, \quad f(m,n) = mn+2$
        (iii) $f : \mathbb{N}_0^2 \to \mathbb{N}_0\,, \quad f(m,n) = m^{n+1}$
    (c) Describe a function $g : \mathbb{N}_0 \to \mathbb{N}_0$ which is not recursive. Justify your answer.
    (You may assume that there exists a list $f_1, f_2, \cdots$, which includes all recursive (partial) functions, possibly with repetitions, and which may include partial functions that are not defined anywhere.)